Definitions

Definite integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into subintervals of length $\Delta x = \frac{b-a}{n}$ and choose x_i^* from each interval.

Then $\int_a^b f(x)\ \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.

Antiderivative: An anti-derivative of $f(x)$ is a function $F(x)$ such that $F' = f$.

Indefinite integral: $\int f(x)\ \mathrm{d}x = F(x) + C$, where F is an anti-derivative of f.

FTC ("integration and differentiation are inverse processes")

Part 1: $\frac{d}{dx}\int_a^x f(t)\ \mathrm{d}t = f(x)$.

Know how to apply the chain rule with part 1!

Part 2: $\int_a^b F'(x)\ \mathrm{d}x = F(b) - F(a)$

Main application of FTC2: integrating the derivative of F tells us the net change in $F(x)$ from $x = a$ to $x = b$.

eg, $\int_{t_1}^{t_2} v(t)dt$ = net distance traveled = net change in position from time t_1 to t_2 (*not* total distance traveled (in general))

Applications

Area between curves: The formulas for the two main cases are:

$\int_a^b$[top function] − [bottom function] dx and $\int_c^d$[right function] − [left function] dy

Volume: We can find the volume of a solid by adding up areas of cross sections of the solid. The main formula is $\int_a^b A(x)\ \mathrm{d}x$ or $\int_c^d A(y)\ \mathrm{d}y$ where $A(x), A(y)$ give the area of a cross section of the solid. The two main cases are:

Disks/Washers: $A = \pi\big((\text{outer radius})^2 - (\text{inner radius})^2\big)$. Cross sections are perpendicular to the axis of rotation.

Cylindrical shells: $A = 2\pi(\text{radius})(\text{height})$. Cross sections are parallel (she||s) to the axis of rotation.

Average value: $f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\ \mathrm{d}x$ = average value of $f(x)$ for $a \le x \le b$.

Work = Force × Distance

Method I: Distance in pieces: Chop up the distance and add up the work required to move each tiny distance $\Delta x \Rightarrow W = \int_a^b \text{force}\ dx$.

Method II: Object in pieces: Chop up the object and add up the work required to move each piece the *whole* distance $\Rightarrow W = \int_a^b \text{force} \times \text{distance}\ dx$.

Hooke's Law: Force required to stretch a spring x units beyond natural length proportional to $x : f(x) = kx$.

Useful formulas: Force = mass × acceleration and density= $\frac{\text{mass}}{\text{volume}}$

Note: Pounds=unit of force and Kg= unit of mass

Arc length

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$ if $y = f(x), a \le x \le b$.

$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy$ if $x = g(y), c \le y \le d$.

Arc length function: $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}dt$ = length of arc from the point $(a, f(a))$ to $(x, f(x))$.

Surface area of a solid of revolution
Rotation about x-axis: $S = 2\pi \int y \; \mathrm{d}s$,
Rotation about y-axis: $S = 2\pi \int x \; \mathrm{d}s$,
where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ if $y = f(x), a \le x \le b$.
$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$ if $x = g(y), c \le y \le d$.

Center of mass
Let ρ be the uniform density of a plate that is the region bounded by the curves $f(x)$ and $g(x)$, where $f(x) \ge g(x)$.

Moments M_x and M_y: measure the tendency of a region to rotate about the x- and y-axis, respectively:
$M_x = \rho \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx$, $M_y = \rho \int_a^b x(f(x) - g(x))dx$.

Center of mass: Let $A = \int_a^b f(x) - g(x) dx$ be the area of the plate and $M = \rho \times A$ be the mass of the plate. Then the coordinates of the center of mass $(\overline{x}, \overline{y})$ are:
$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x(f(x) - g(x))dx}{A}, \text{ and } \overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx}{A}$$

Integration techniques

u-substitution: works for integrating compositions of functions; pick u to be the 'inside' function.

Integration by parts - undoing the product rule: $\int u \; \mathrm{d}v = uv - \int v \; \mathrm{d}u$.
Generally, picking u in this descending order works:
Inverse trig
Logarithm
Algebraic (polynomial)
Trig
Exponential

Partial fractions: -
If necessary, make a substitution to get a ratio of polynomials
If the degree of the numerator is $\ge$ the degree of denominator, do long division first.
Then factor the denominator into linear terms and irreducible quadratics.

factor in denominator		term in partial fraction decomposition
$(ax+b)^k$	$\Rightarrow$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_k}{(ax+b)^k}$
$(ax^2+bx+c)^k$	$\Rightarrow$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ... + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Integral Tables: see back of textbook. Often you will need to make a u-substitution first (u = inside function) to be able to apply a formula. Less often you'll "complete the square": eg: $x^2 + 6x + 5 = x^2 + 6x + 9 - 9 + 5 = (x+3)^2 - 4$ (divide x coefficient by 2, square it, and add and subtract it. Note: works when coefficient of x^2 is 1)

Improper integrals

Type 1: infinite interval: $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$, $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$

Type 2: discontinuity in interval: -

f discontinuous at a: $\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$

f discontinuous at b: $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$

f discontinuous at $c, a < c < b$: $\int_a^b f(x)dx = \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx$

Comparison Test: If f, g are continuous with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then:
(a) $\int_a^\infty f(x)dx$ convergent $\Rightarrow \int_a^\infty g(x)dx$ convergent (if a larger function f converges, so does g)
(b) $\int_a^\infty g(x)dx$ divergent $\Rightarrow \int_a^\infty f(x)dx$ divergent (if a smaller function g diverges, so does f)
Note the comparison test doesn't help if a smaller function converges, or if a larger function diverges.

Approximate integration:

Areas under curves: Choose n = number of rectangles and choose x_i^* from each interval.
Then $\int_a^b f(x)\ \mathrm{d}x \approx \sum_{i=1}^{n} f(x_i^*)\Delta x = \Delta x\big[f(x_1^*) + f(x_2^*) + ... + f(x_n^*)\big]$, where $\Delta x = \frac{b-a}{n}$.
Commonly x_i^* is chosen to be the right endpoint, left endpoint, or midpoint.

Other regions: Know how to approximate areas of regions between curves and volumes of revolution using either disks or cylindrical shells.

Upper/lower bounds:
For an increasing function, using left endpoints gives a lower bound and using right endpoints gives an upper bound.
For a decreasing function, using right endpoints gives a lower bound and using left endpoints gives an upper bound.

Trapezoidal rule: $\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)\right]$
Error bound: If $|f''(x)| \leq K$ on $[a, b]$ then $\left|\int_a^b f(x)dx - T_n\right| \leq \frac{K(b-a)^3}{12n^2}$

Midpoint rule: $\int_a^b f(x)dx \approx M_n = \Delta x\left[f(x_1^*) + ... + f(x_n^*)\right]$ where $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$
Error bound: If $|f''(x)| \leq K$ on $[a, b]$ then $\left|\int_a^b f(x)dx - M_n\right| \leq \frac{K(b-a)^3}{24n^2}$

Simpson's rule (n even): -
$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$
Error bound: If $|f^{(4)}(x)| \leq K$ on $[a, b]$ then $\left|\int_a^b f(x)dx - S_n\right| \leq \frac{K(b-a)^5}{180n^4}$